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Combinatorial properties of a general domination problem with parity constraints

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Abstract

We consider various properties of a general parity domination problem: given a graph G on n vertices, one is looking for a subset S of the vertex set such that the open/closed neighborhood of each vertex contains an even/odd number of vertices in S (it is prescribed individually for each vertex which of these applies). We define the parameter $s(G)$ to be the number of solvable instances out of 4^n possibilities and study the properties of this parameter. Upper and lower bounds for general graphs and trees are given as well as a remarkable recurrence formula for rooted trees. Furthermore, we give explicit formulas in several special cases and investigate random graphs.

Keywords: Domination problem; Parity constraint; Matrix algebra

1. Introduction

The classical problem of domination asks for a subset S of the vertex set of a graph $G = (V(G), E(G))$ of minimum cardinality such that $N[v] \cap S \neq \emptyset$ for all $v \in V(G)$, where $N[v] = \{u \in V(G) \mid \exists e = (u, v) \in E(G)\} \cup \{v\}$ denotes the closed neighborhood of v. Quite a lot of different modifications and generalizations of this problem are known. For instance, the *k-tuple domination problem* [16] asks for a minimum set S such that $|N[v] \cap S| \geq k$ for all vertices v. Similarly, in the *k-domination problem* [9,10] the task is to find a set S of minimum cardinality such that $|N(v) \cap S| \geq k$ for all vertices v, where $N(v) = \{u \in V(G) \mid \exists e = (u, v) \in E(G)\}$ denotes the open neighborhood of v. Even more generally, one can prescribe a set R_v for every vertex v and pose the question whether there exists a set S such that $|N[v] \cap S| \in R_v$ (or $|N(v) \cap S| \in R_v$) for all vertices v.

The special cases $R_v = \{1, 2, 3, \ldots\}$ and $R_v = \{k, k + 1, \ldots\}$ have already been mentioned. These and other variants, such as $R_v = \{1\}$, are discussed in the book of Haynes, Hedetniemi and Slater [17]. Another interesting kind of domination problem involves parity constraints. It has been treated in a series of papers [1–4,8], motivated by the following remarkable result of Sutner [19]:

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Theorem 1 (*Sutner [19]*). *For every graph $G = (V(G), E(G))$, there exists a set $S \subseteq V(G)$ such that $|N[v] \cap S|$ is odd for every $v \in V(G)$.*

This means that the domination problem for $R_v = \{1, 3, 5, \ldots\}$ is always solvable if we consider closed neighborhoods. Thus, it is natural to consider a general parity assignment problem, where each R_v is either $\{1, 3, 5, \ldots\}$ or $\{0, 2, 4, \ldots\}$. It has been treated quite extensively in [1–4], where the notions of "parity dimension" and "all parity realizable graphs" have been introduced. Wagner [21] gives a recursive procedure for determining the parity dimension of a tree, which is then applied to enumeration problems involving the parity dimension, in particular to counting all parity realizable trees. All these papers deal with parity domination with respect to closed neighborhoods, but analogous results exist for the case of open neighborhoods as well—see for instance [14] and the references therein. Other works such as [15,20] study parity domination with a focus on complexity results.

In another recent paper, Gassner and Hatzl [13] discuss an even more general parity domination problem from an algorithmic point of view: for every vertex v, we impose exactly one of the following four constraints:

- $|N(v) \cap S| \equiv 0 \bmod 2$,
- $|N(v) \cap S| \equiv 1 \bmod 2$,
- $|N[v] \cap S| \equiv 0 \bmod 2$,
- $|N[v] \cap S| \equiv 1 \bmod 2$,

i.e., the open/closed neighborhood has to contain an even/odd number of vertices in S. One of the reasons to consider domination problems with parity constraints lies in the fact that the problem can be stated easily in terms of matrix algebra: in the following, we denote by A and $A + I$ the open neighborhood matrix (adjacency matrix) and the closed neighborhood matrix respectively (I is the identity matrix). Furthermore, we use a vector $a \in \{0, 1\}^{V(G)}$ as a representation for the neighborhood information (i.e., whether the open or closed neighborhood is considered for a certain vertex): If the entry a_v that corresponds to a vertex v is 0, the open neighborhood is of interest for this vertex, and the closed neighborhood otherwise. Another vector $b \in \{0, 1\}^{V(G)}$ represents the prescribed parities. Using these vectors, our requirements can be written as

$$(A + \mathrm{diag}(a))x = b \tag{1}$$

over the field $\mathbb{F}_2$. Obviously, $x_v = 1$ if and only if $v \in S$.

In this paper, we are interested in the number of solvable instances—a parameter that plays an analogous role to the parity dimension (for the domination problem with parity constraints considering closed subsets only): let the *solvability number* $s(G)$ denote the number of solvable instances for a graph G, i.e., the number of pairs $(a, b) \in \{0, 1\}^{V(G)} \times \{0, 1\}^{V(G)}$ such that there exists a vector x satisfying the system of linear equations in (1).

Basic linear algebra gives us the following simple lemma:

Lemma 2. *Let $G = (V(G), E(G))$ be a graph, then*

$$s(G) = \sum_{a \in \{0,1\}^{V(G)}} 2^{\mathrm{rk}(A+\mathrm{diag}(a))}, \tag{2}$$

where $\mathrm{rk}(B)$ denotes the rank of a matrix B over $\mathbb{F}_2$.

Remark 3. Replacing 2 by a variable x in the above formula, we obtain a polynomial

$$S_G(x) = \sum_{a \in \{0,1\}^{V(G)}} x^{\mathrm{rk}(A+\mathrm{diag}(a))}$$

with interesting properties: $S_G(0) = 1$ if G is the empty graph, and $S_G(0) = 0$ otherwise; $S_G(1) = 2^{|V(G)|}$, and $\frac{S'_G(1)}{S_G(1)}$ gives the average rank of $A + \mathrm{diag}(a)$, as a varies over all possible vectors.

Corollary 4.

$$2^{|V(G)|} \le s(G) \le 4^{|V(G)|}.$$

Thus, $\frac{\log_2 s(G)}{|V(G)|}$ can be seen as a "normalisation" of $s(G)$ which always lies between 1 and 2. In Section 2, we will improve these bounds for arbitrary graphs. Moreover, explicit formulas for some graphs are given. Section 3 is dedicated to trees and a recurrence formula for rooted trees. This recursion enables us to improve the lower bound for trees. The last section deals with the expected value of $s(G)$ for random graphs.

2. Special cases and inequalities

In this section, explicit formulas for special graphs are deduced. All these formulas are obtained using Eq. (2) and arguments from linear algebra.

Proposition 5. *The solvability numbers of the empty graph and the complete graph on n vertices are* 3^n *and* $2 \cdot 3^n - 5 \cdot 2^{n-2} + (-2)^{n-2}$ *respectively.*

Proof. The formula for the empty graph is trivial: just note that the rank of $A + \mathrm{diag}(a)$ equals the number of 1's in a (since $A = 0$), from which the formula follows from the binomial theorem.

The formula for the complete graph is slightly trickier: if $a \neq 0$, one of the rows of $A + \mathrm{diag}(a)$ consists entirely of 1's. We subtract this row from all others and obtain a matrix whose rows (except one) contain at most one 1. It is easy to see now that the rank is the number of 0's in a, increased by 1. Applying the binomial theorem again yields the main term $2 \cdot 3^n$. Since the rank of the adjacency matrix A is n if n is even and $n - 1$ if n is odd (by a similar argument—note that the vector $(1, 1, \ldots, 1)$ is spanned by the rows of A if and only if n is even), we obtain $2 \cdot 3^n - 2^n$ if n is even and $2 \cdot 3^n - 3 \cdot 2^{n-1}$ if n is odd, which reduces to $2 \cdot 3^n - 5 \cdot 2^{n-2} + (-2)^{n-2}$ for all n. □

Proposition 6. *The solvability numbers of the path* P_n *and the cycle* C_n *on n vertices are*

$$s(P_n) = \frac{5}{6} \cdot 4^n + \frac{1}{6} \cdot (-2)^n \tag{3}$$

and

$$s(C_n) = \frac{5}{8} \cdot 4^n - \frac{1}{4} \cdot (-2)^n, \tag{4}$$

respectively.

Proof. Again, we want to determine the number of vectors a for which $A + \mathrm{diag}(a)$ has a specific rank. For this purpose, we only have to find nontrivial solutions x to the equation $(A + \mathrm{diag}(a))x = 0$. Let us start with the path P_n, and let its vertices be $v_1, v_2, \ldots, v_n$, where v_1 and v_n are the leaves. The corresponding entries of a and x are denoted by $a_1, a_2, \ldots, a_n$ and $x_1, x_2, \ldots, x_n$. Suppose that x_1 is given. Then x_2 is uniquely defined by the equation $a_1x_1 + x_2 = 0$, x_3 is uniquely defined by $x_1 + a_2x_2 + x_3 = 0$, and so on. Hence, a solution to $(A + \mathrm{diag}(a))x = 0$ is uniquely determined by x_1 (if it exists). It is plain that $x_1 = 0$ always leads to the trivial solution, so we may assume $x_1 = 1$. In order to determine whether a nontrivial solution exists for a given vector a, we can use a finite automaton that reads the entries $a_1, a_2, \ldots$ of a (in this order). A state is given by two consecutive entries x_i, x_{i+1} of x. The initial state ($i = 0$) is $(0, 1)$ (0 for the non-existing vertex v_0 and 1 by the assumption that $x_1 = 1$). The general equation

$$x_{i-1} + a_ix_i + x_{i+1} = 0$$

implies that the state transitions are

$$\begin{aligned} a_i &= 0 : (0, 1) \to (1, 0), (1, 0) \to (0, 1), (1, 1) \to (1, 1), \\ a_i &= 1 : (0, 1) \to (1, 1), (1, 0) \to (0, 1), (1, 1) \to (1, 0). \end{aligned}$$

Fig. 1 shows the resulting automaton. A nontrivial solution exists if and only if the final state is $(x_n, x_{n+1}) = (1, 0)$. If a nontrivial solution exists, it is unique.

Now it is an easy exercise to determine the number of vectors a for which a nontrivial solution exists (and for which the rank of $A + \mathrm{diag}(a)$ is thus $n - 1$ rather than n) from the adjacency matrix of the automaton, which in turn yields the formula for $s(P_n)$: the number of vectors for which $A + \mathrm{diag}(a)$ has full rank n is given by $\frac{2}{3} \cdot 2^n + \frac{1}{3} \cdot (-1)^n$, whereas the number of vectors for which $A + \mathrm{diag}(a)$ has rank $n - 1$ is given by $\frac{2}{3} \cdot 2^{n-1} + \frac{1}{3} \cdot (-1)^{n-1}$.

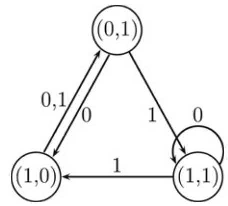

Fig. 1. An automaton used for determining the rank of $A + \mathrm{diag}(a)$ in the case of the path.

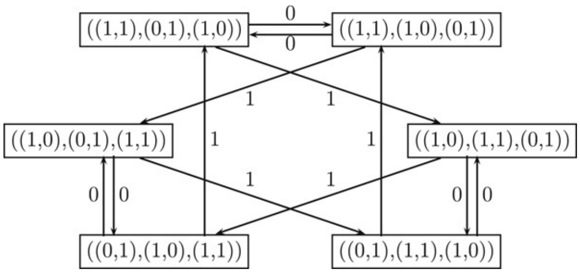

Fig. 2. An automaton used for determining the rank of $A + \mathrm{diag}(a)$ in the case of the cycle.

The procedure is basically the same for the cycle. However, we have to prescribe two initial values x_1 and x_2, for which there are four possibilities. $(x_1, x_2) = (0, 0)$ leads to the trivial solution again. The automaton is the same as for the path—a nontrivial solution exists if and only if $(x_1, x_2) = (x_{n+1}, x_{n+2})$ for some initial values (x_1, x_2) in the set $\{(0, 1), (1, 0), (1, 1)\}$. If a nontrivial solution exists for exactly one of them, the rank of $A + \mathrm{diag}(a)$ is $n - 1$. However, if a nontrivial solution exists for all three, the rank is $n - 2$. To take account of this, we also count the number of vectors a for which this occurs. This is done by means of another automaton, whose states are triples of the states of the first automaton (Fig. 2). The automaton returns to its initial state after n steps if and only if there is a nontrivial solution for all three choices of (x_1, x_2).

Now it turns out that the rank of $A + \mathrm{diag}(a)$ is n for $\frac{2}{3} \cdot 2^{n-1} + \frac{1}{3} \cdot (-1)^{n-1}$ vectors, $n - 1$ for 2^{n-1} vectors and $n - 2$ for $\frac{2}{3} \cdot 2^{n-2} + \frac{1}{3} \cdot (-1)^{n-2}$ vectors, which yields the formula for $s(C_n)$. $\square$

A formula for the star will be given in Section 3. Similarly, more tedious calculations yield the following formulas:

Proposition 7. *The solvability numbers of the fan F_n and the wheel W_n on n vertices are*

$$s(F_n) = \frac{35}{48} \cdot 4^n - \frac{1}{24} \cdot (-2)^n + \frac{5}{4\sqrt{-7}} \left(\left(-1 + \sqrt{-7}\right)^{n-1} - \left(-1 - \sqrt{-7}\right)^{n-1} \right)$$

and

$$s(W_n) = \frac{85}{128} \cdot 4^n + \frac{15}{64} \cdot 2^n + \frac{7}{32} \cdot (-2)^n - \frac{25}{32} \left(\left(-1 + \sqrt{-7}\right)^{n-1} + \left(-1 - \sqrt{-7}\right)^{n-1} \right),$$

respectively.

Proof. We concentrate on the fan, the proof for the wheel being essentially the same. Let the vertices be $v_0, v_1, \ldots, v_{n-1}$, where $v_1, v_2, \ldots, v_{n-1}$ form a path and v_0 is adjacent to all other vertices. The corresponding entries of a and x are denoted by $a_0, a_1, \ldots, a_{n-1}$ and $x_0, x_1, \ldots, x_{n-1}$ again. We will distinguish the two cases $x_0 = 0$ and $x_0 = 1$. In the former case,

$$x_{i-1} + a_i x_i + x_{i+1} = 0,$$

and we use essentially the same automaton as for the path, but our states are now triples $(x_i, x_{i+1}, \sum_{j=1}^{i} x_j)$ (of course, the sum is taken modulo 2). There are two possible initial triples, namely $(0, 0, 0)$ and $(0, 1, 0)$, where the former leads to the trivial solution. The final triple has to be either $(0, 0, 0)$ (which is the case for the trivial solution)

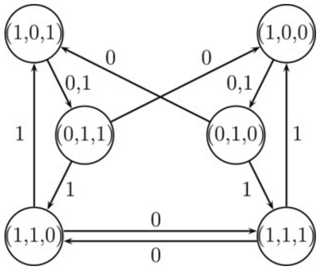

Fig. 3. The first automaton used for determining the rank of $A + \mathrm{diag}(a)$ in the case of the fan.

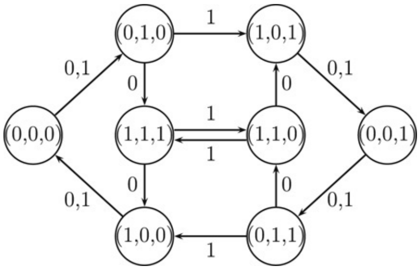

Fig. 4. The second automaton used for determining the rank of $A + \mathrm{diag}(a)$ in the case of the fan.

or $(1, 0, 0)$, since we have the condition

$$a_0 x_0 + \sum_{j=1}^{n-1} x_j = 0.$$

The automaton is shown in Fig. 3. Similarly, we get an automaton with eight states (see Fig. 4) in the case $x_0 = 1$. The initial state is either $(0, 0, 0)$ or $(0, 1, 0)$, whereas the terminal state is either $(0, 0, 0)$ or $(1, 0, 0)$ if $a_0 = 0$ and either $(0, 0, 1)$ or $(1, 0, 1)$ if $a_0 = 1$. In all the cases, we can easily determine the number of possibilities for $a_1, a_2, \ldots$ such that there is a nontrivial solution with $x_1 = 0$ ($x_1 = 1$, respectively). If there is a solution to one of these, we know that the rank of $A + \mathrm{diag}(a)$ is $\leq n - 1$. Furthermore, this matrix has rank $n - 2$ if and only if there are nontrivial solutions with $(x_0, x_1) = (0, 1)$, $(x_0, x_1) = (1, 0)$ and $(x_0, x_1) = (1, 1)$. Indeed, it is sufficient that there are solutions with $x_0 = 1$ and both $x_1 = 0$ and $x_1 = 1$. We use the same strategy as in the proof of Proposition 6, i.e., we construct a larger automaton whose states are pairs of triples $(x_i, x_{i+1}, \sum_{j=1}^{i} x_j)$. The automaton is too large to be depicted here, but it is easy to construct it as well as its adjacency matrix from the automaton in Fig. 4 with the help of a computer. As in the proof of Proposition 6, we find that the matrix $A + \mathrm{diag}(a)$ has

- rank n for $2^{n-1} + \frac{1}{\sqrt{-7}}\left(\frac{-1+\sqrt{-7}}{2}\right)^{n-1} - \frac{1}{\sqrt{-7}}\left(\frac{-1-\sqrt{-7}}{2}\right)^{n-1}$ different vectors a,
- rank $n-1$ for $\frac{5}{6}2^{n-1} + \frac{1}{6}(-1)^{n-1} - \frac{1}{2\sqrt{-7}}\left(\frac{-1+\sqrt{-7}}{2}\right)^{n-1} + \frac{1}{2\sqrt{-7}}\left(\frac{-1-\sqrt{-7}}{2}\right)^{n-1}$ different vectors a,
- rank $n-2$ for $\frac{1}{6}2^{n-1} - \frac{1}{6}(-1)^{n-1} - \frac{1}{2\sqrt{-7}}\left(\frac{-1+\sqrt{-7}}{2}\right)^{n-1} + \frac{1}{2\sqrt{-7}}\left(\frac{-1-\sqrt{-7}}{2}\right)^{n-1}$ different vectors a,

and the formula for $s(F_n)$ follows. $\square$

Theorem 8. *Let E_n be the empty graph on n vertices. Then,*

$$s(G) \geq s(E_n)$$

holds for all graphs G with $|V(G)| = n$. Moreover, the inequality is strict if $G \neq E_n$.

Proof. First we prove that there is always a vector a_0 such that $A + \mathrm{diag}(a_0)$ has full rank. This is done by means of induction. The statement is trivial for $n = 1$. For the induction step, let v be the vertex that corresponds to the last row of A. By the induction hypothesis, we can choose a vector a_0' such that $A' + \mathrm{diag}(a_0')$, where A' is the adjacency matrix of $G \setminus \{v\}$, is invertible. Hence, there is a unique set R of rows of $A' + \mathrm{diag}(a'_0)$ whose sum is equal to the last row of A (without the last element). If there is an even number of neighbors of v among the vertices corresponding to R, we set $b = 1$, otherwise we set $b = 0$. Now, we obtain a vector a_0 by appending b to a'_0. Then, $A + \mathrm{diag}(a_0)$ has full rank again: by our choice of b, the last row is linearly independent of the others, which in turn are linearly independent by the assumption that $A' + \mathrm{diag}(a'_0)$ is invertible. This finishes the induction.

Therefore, there is always at least one vector a_0 for which $A + \mathrm{diag}(a_0)$ has full rank. Since changing an entry of a changes the rank at most by 1, one has $\mathrm{rk}(A + \mathrm{diag}(a_0) + \mathrm{diag}(a)) \geq \mathrm{rk}(\mathrm{diag}((1, 1, \dots, 1) + a))$ for all $a \in \{0, 1\}^n$. Thus, $s(G) \geq s(E_n)$ follows. Due to the fact that the empty graph is the only graph for which the rank can be 0, the bound is strict if $G \neq E_n$. □

Similarly, an upper bound for $s(G)$ is given in the next theorem.

Theorem 9. *Let P_n be the path with n vertices. Then,*

$$s(G) \leq s(P_n)$$

holds for all graphs G with $|V(G)| = n$. Moreover, the inequality is strict if $G \neq P_n$.

Proof. In the proof of Proposition 6 it was shown that $|\{a \in \{0, 1\}^n : \mathrm{rk}(A + \mathrm{diag}(a)) < n\}| = \frac{2}{3}2^{n-1} + \frac{1}{3}(-1)^{n-1}$ where A is the adjacency matrix of P_n. Moreover, it was proved that $\mathrm{rk}(A + \mathrm{diag}(a))$ is n or $n - 1$ for all $a \in \{0, 1\}^n$. Thus, it suffices to show that

$$|\{a \in \{0, 1\}^n : \mathrm{rk}(A + \mathrm{diag}(a)) < n\}| \geq c(n), \tag{5}$$

where

$$c(n) = \frac{2}{3}\, 2^{n-1} + \frac{1}{3}\, (-1)^{n-1},$$

holds for all symmetric $n \times n$ matrices A. Without loss of generality, we can restrict ourselves to adjacency matrices, i.e., to matrices where all diagonal entries are 0. In order to prove this we use induction on n. It is easy to see that the inequality given in (5) holds for $n = 1$ and $n = 2$. Moreover, if $n = 2$ we have equality if and only if A is the adjacency matrix of P_2.

Let A be an $(n + 1) \times (n + 1)$ adjacency matrix, and let the corresponding graph be G. The vertices are denoted by $v_1, v_2, \dots, v_{n+1}$. Furthermore, we assume that (5) holds for n and $n - 1$. For the inductive step, we give lower bounds on the cardinality of the following two sets:

$$S_1 = \{a \in \{0, 1\}^{n+1} : \mathrm{rk}(A + \mathrm{diag}(a)) < n + 1 \text{ and } a_1 = 1\}, \tag{6}$$

$$S_2 = \{a \in \{0, 1\}^{n+1} : \mathrm{rk}(A + \mathrm{diag}(a)) < n + 1 \text{ and } a_1 = 0\}. \tag{7}$$

Let us assume we are given a symmetric matrix $A + \mathrm{diag}(a)$ with $a_1 = 1$. In order to calculate its rank, we add the first row to all rows i with $a_{i,1} = 1$. Then, we obtain a matrix $\tilde{A}$ where $\tilde{a}_{i,1} = 0$ holds for all $i = 2, \dots, n + 1$ and $\tilde{a}_{1,1} = 1$. Note also that $\mathrm{rk}(\tilde{A}) = \mathrm{rk}(A + \mathrm{diag}(a))$.

Consequently, the rank of $\tilde{A}$ equals 1 plus the rank of its $(1, 1)$-minor, i.e., the matrix that results from deleting the first row and first column. Note that this minor is still a symmetric matrix (since $a_{i,1}a_{1,j}$ is added to each entry $a_{i,j}$, which is clearly symmetric), so (by the induction hypothesis) there are at least $c(n)$ different diagonal vectors for which it does not have full rank. Moreover, each of the possible diagonals appears exactly once as a varies over all possible vectors. Thus, by the induction hypothesis, $|S_1| \geq c(n)$ follows.

The lower bound on the cardinality of S_2 can be obtained similarly. We assume without loss of generality that $a_{2,1} = a_{1,2} = 1$. Otherwise, all entries in the first row, respectively column, are 0 and we can conclude that $|S_2| = 2^n$. Then, Eq. (5) follows immediately for $n + 1$, because $2^n + c(n) \geq c(n + 1)$. Now we use some elementary row and column operations again to calculate the rank of $A + \mathrm{diag}(a)$. First we add the second row to all rows i with $a_{i,1} = 1$. Afterwards, we add the second column to all columns j with $b_{1,j} = 1$. After interchanging the first and the second row, we end up with a matrix $\overline{A}$ for which $\overline{a}_{1,1} = \overline{a}_{2,2} = 1$ and $\overline{a}_{i,1} = 0$ for all $i = 2, \dots, n + 1$ and $\overline{a}_{2,j} = 0$ for all $j = 3, \dots, n + 1$. Note that $\mathrm{rk}(\overline{A}) = \mathrm{rk}(A + \mathrm{diag}(a))$ and that $\overline{A}$ has full rank if and only if the matrix obtained from $\overline{A}$ by deleting the first two rows and columns has full rank. Let us denote this $(n - 2) \times (n - 2)$ matrix by $\overline{A}'$. Note that $\overline{A}'$ is also symmetric, since $a_{i,1}a_{2,j}+a_{1,j}a_{i,2}+a_{i,1}a_{1,j}a_2$ is added to each entry $a_{i,j}$, which is symmetric in i and j again. Furthermore, each of the possible diagonals of $\overline{A}'$ appears exactly twice as a varies over all possible vectors.

Making use of the induction hypothesis, we can conclude that $|S_2| \geq 2c(n-1)$. Summarizing, we get

$$|\{a \in \{0,1\}^{n+1} : \mathrm{rk}(A + \mathrm{diag}(a)) < n+1\}| = |S_1| + |S_2| \geq c(n) + 2c(n-1) = c(n+1), \tag{8}$$

which completes the induction step for the inequality. Obviously, equality holds if G is a path, and so it remains to show that there is no other possibility. Note that by the preceding arguments and the induction hypothesis, equality can only hold if the $(1,1)$-minor of the matrix $\tilde{A}$ is the adjacency matrix of a path. If $N = N(v_1)$ is the set of neighbors of v_1 in G, then this implies that G is constructed as follows:

- Let $v_2, v_3, \ldots, v_{n+1}$ form a path P.
- Delete all edges between vertices in N that are adjacent in P.
- Add all edges between vertices in N that are not adjacent in P.
- Connect v_1 to all vertices in N.

If $|N| = \deg v_1 = k$ and l is the number of edges between vertices in N in the path P (note that $l \leq k-1$ unless $k=0$), this implies that

$$|E(G)| = n - 1 + k + \binom{k}{2} - 2l. \tag{9}$$

Without loss of generality, we may assume $0 < k < n$ (otherwise, G is either the empty or the complete graph, and we have already shown that $s(E_n) < s(K_n) < s(P_n)$ for $n \geq 3$). Then at least one of the vertices $v_2, v_3, \ldots, v_{n+1}$ is not an element of N, and it follows that this vertex has degree 1 or 2. Repeating the argument for this vertex instead of v_1, we obtain $|E(G)| = n$ or $|E(G)| = n+2$. On the other hand,

$$|E(G)| = n - 1 + k + \binom{k}{2} - 2l \geq n - 1 + k + \binom{k}{2} - 2(k-1) = n + \frac{(k-1)(k-2)}{2},$$

and it follows that $k \leq 3$. But $k = 3$ implies $|E(G)| \in \{n+1, n+3, n+5\}$, a contradiction. Therefore, $k \in \{1,2\}$. By the above arguments, there can be no vertex of degree 3 or 4, which leaves us with only two possibilities: either $k = 1$, and v_1 is attached to one of the endpoints of P, or $k = 2$ and v_1 is attached to two adjacent vertices of P (and the edge between them deleted). In either case, G is a path. □

Theorem 10. *Let G be a graph and G^c its complement. Then we have*

$$\frac{1}{2}s(G) \leq s(G^c) \leq 2s(G).$$

The constant 2 is the best possible in this inequality.

Proof. Let A, A' be the adjacency matrices of G and G^c. Furthermore, for a vector a, let $a' = e - a$, where e is the vector whose entries are all 1 (i.e., all entries of a are switched). Finally, let E be the $|V(G)| \times |V(G)|$ matrix whose entries are all 1. Then, it is clear that

$$A' + \mathrm{diag}(a') = E - (A + \mathrm{diag}(a)).$$

Furthermore, if we add the vector e as a row to some matrix M and $E - M$, the space that is spanned by the rows of the two matrices becomes the same, and the rank increases at most by one. Hence, the ranks of M and $E - M$ differ at most by one. Applying this to $A + \mathrm{diag}(a)$ and summing over all a, we have, by Lemma 2,

$$\frac{1}{2}s(G) \leq s(G^c) \leq 2s(G),$$

as claimed. The formulas for the empty graph and the complete graph show that the constant 2 is indeed the best possible. □

3. A property of distinguished vertices and a recursion for rooted trees

Let v be a distinguished vertex (the "root") of a graph G. Given the neighborhood vector a and the parity vector b, we say that an instance is "solvable" if there exists a solution x in (1) and "almost solvable" if there exists a solution x for the instance (a, b'), where b' results from replacing the entry b_v belonging to v by $1 - b_v$. Furthermore, if a solution vector x is given, we say that we "use the root" if the entry x_v that corresponds to v is 1, and that we "don't use the root" otherwise. Now, the following lemma holds, which proves to be extremely useful in the treatment of trees:

Lemma 11. *Given a root $v \in V(G)$, a neighborhood vector a and a parity vector b, the following situations (and no others) are possible:*

(1) *the instance is neither solvable nor almost solvable,*
(2) *the instance is solvable using the root or without using the root, but not almost solvable,*
(3) *the instance is almost solvable using the root or without using the root, but not solvable,*
(4) *the instance is solvable using the root and almost solvable without using the root,*
(5) *the instance is almost solvable using the root and solvable without using the root,*
(6) *the instance is solvable and almost solvable if we use the root, but neither solvable nor almost solvable if we don't use it,*
(7) *the instance is solvable and almost solvable if we don't use the root, but neither solvable nor almost solvable if we use it.*

Furthermore, the number of instances belonging to case (2) *is the same as the number of instances belonging to case* (3)*, the number of instances belonging to case* (4)*, and the number of instances belonging to case* (5)*. Similarly, the number of instances belonging to case* (6) *is the same as the number of instances belonging to case* (7).

Proof. Set $M = A + \mathrm{diag}(a)$. Since the kernel and the image of a symmetric matrix are orthogonal, we know that precisely one of the following possibilities holds:

- there is a vector x with $x_v = 1$ such that $Mx = 0$,
- there is a vector x such that $Mx = e_v$, where e_v is the unit vector whose entry is 1 at the position that corresponds to v and 0 otherwise.

It is easy to see that the first statement is equivalent to the fact that all solvable or almost solvable instances (a, b) can be solved (almost solved) with or without using the root. On the other hand, $Mx = e_v$ is equivalent to the fact that every instance that can be solved can also be almost solved and vice versa. Hence, if the first statement holds, all (almost) solvable instances (a, b) belong to case (2) or (3), and there are no instances that can be solved as well as almost solved. Whether an instance belongs to case (2) or (3) only depends on b_v, and thus the number of instances is the same for each of these two cases.

If the second statement holds, all (almost) solvable instances (a, b) belong to case (4) or (5) (depending only on b_v) if $x_v = 1$, and to case (6) or (7) if $x_v = 0$. In the former case, note that x becomes a solution to $Mx = 0$ if a_v is changed from 0 to 1 or vice versa. Hence, the instances belonging to cases (2) and (3) and the instances belonging to cases (4) and (5) are equinumerous.

Finally, we have to prove the fact that there are equally many instances belonging to cases (6) and (7). But this is also obvious, since they form affine spaces of the same dimension which differ only by a column of M. □

In the following, we will denote the number of instances belonging to case (2) by $x(G, v)$ whereas for the number of instances belonging to case (6) the term $y(G, v)$ is used. Using the above proposition $s(G) = 3x(G, v) + 2y(G, v)$ follows.

The following proposition provides a recursion for the auxiliary parameters $x(G, v)$ and $y(G, v)$ in the case of rooted trees, which enables us to deal with all sorts of questions concerning the solvability number of trees:

Proposition 12. *Let T be a rooted tree and v its root. Furthermore, let the branches of T be $T_1, \ldots, T_k$ and their roots (the neighbors of v) $v_1, \ldots, v_k$. Then the identities*

$$x(T, v) = \prod_{i=1}^{k} (2x(T_i, v_i) + 2y(T_i, v_i))$$

and

$$y(T,v)=4\left(\prod_{i=1}^{k}(3x(T_i,v_i)+2y(T_i,v_i))-\prod_{i=1}^{k}(2x(T_i,v_i)+2y(T_i,v_i))\right)$$

hold, which can be simplified as follows:

$$s(T)=8\prod_{i=1}^{k}s(T_i)-5\prod_{i=1}^{k}t(T_i,v_i),$$

$$t(T,v)=8\prod_{i=1}^{k}s(T_i)-6\prod_{i=1}^{k}t(T_i,v_i),$$

where $t(T,v)=2x(T,v)+2y(T,v)$ is another auxiliary parameter.

Proof. Let an instance (a,b) be given and let $(a^{(i)},b^{(i)})$ be the restriction to the i-th branch. Obviously, (a,b) can only be solvable or almost solvable if all restrictions $(a^{(i)},b^{(i)})$ are. We distinguish three cases:

(I) There are two instances $(a^{(i)},b^{(i)})$ and $(a^{(j)},b^{(j)})$ which belong to cases (2) and (3) of Lemma 11 respectively. Then, we must not use the root v in order to solve the subproblem in the i-th subtree, but we have to use it in order to solve the subproblem in the j-th subtree; in view of this contradiction, (a,b) is neither solvable nor almost solvable in this case.

(II) There is at least one instance $(a^{(i)},b^{(i)})$ which belongs to case (2) (or (3)) of Lemma 11. We consider the first possibility, since the second can be treated analogously. Then, we must not use the root v, but we can solve each of the subproblems, and since we may decide in the i-th branch whether we want to use v_i or not, the instance (a,b) is solvable as well as almost solvable in this case. Hence, it belongs to case (6) or (7) of Lemma 11. There are

$$4\cdot 2\left(\prod_{i=1}^{k}(3x(T_i,v_i)+2y(T_i,v_i))-\prod_{i=1}^{k}(2x(T_i,v_i)+2y(T_i,v_i))\right)$$

possible instances for this case (note that there are still 4 ways to choose a_v and b_v!), and since cases (6) and (7) are equinumerous, we obtain the formula for $y(T,v)$.

(III) If there are no instances $(a^{(i)},b^{(i)})$ which belong to case (2) or (3) of Lemma 11, then it follows analogously that (a,b) belongs to one of the cases (2)–(5) (note that we can either use or not use the root v in order to solve the subproblems in this case), the particular case depending on the choice of a_v and b_v. The formula for $x(T,v)$ follows as a simple consequence.

Now, substitution of $s(T_i)=3x(T_i,v_i)+2y(T_i,v_i)$ and $t(T_i,v_i)=2x(T_i,v_i)+2y(T_i,v_i)$ yields

$$x(T,v)=\prod_{i=1}^{k}t(T_i,v_i)$$

and

$$y(T,v)=4\left(\prod_{i=1}^{k}s(T_i)-\prod_{i=1}^{k}t(T_i,v_i)\right),$$

and the formulas for $s(T)$ and $t(T,v)$ follow immediately. □

Making use of this recursion and dynamic programming, the solvability number of a tree can be computed in linear time. We remark that this idea can be generalized (the details, however, are long and tedious) to a larger class, namely graphs with bounded treewidth (see [6]). Note also that the recursive formula provides us with another method to prove the formula (3) stated in Proposition 6 for the path. Moreover, an explicit formula for the star can be obtained:

Corollary 13. *Let S_n be the star on n vertices. Then, the formula $s(S_n)=8\cdot 3^{n-1}-5\cdot 2^{n-1}$ holds.*

Proof. Note that $s(K_1)=3$ and $t(K_1,v)=2$ (where v is the only vertex of K_1). Using this fact the formula is immediately obtained from the above proposition. □

It can even be shown that the star has the smallest number of solvable instances among all trees with a given number of vertices, whereas the path has the largest number of solvable instances (which is clear in view of Theorem 9).

Proposition 14. *Let S_n be the star graph with n vertices. Then,*

$$s(S_n) \leq s(T)$$

holds for all trees $T = (V(T), E(T))$ with $|V(T)| = n$.

Proof. We show using induction on n that $s(S_n) \leq s(T)$ (where $s(S_n) = 8 \cdot 3^{n-1} - 5 \cdot 2^{n-1}$), that $t(T, v)$ attains its minimum value among all trees with n vertices if $T = S_n$ and v is one of its leaves (with a minimum value of $16 \cdot 3^{n-2} - 2^n$), and that $x(T, v) = s(T) - t(T, v)$ is minimal if T is the star and v its center (with a minimum value of 2^{n-1}). This is obvious for $n = 1$ or $n = 2$. For the induction step, note that

$$\begin{aligned} s(T) &= 8\prod_{i=1}^{k} s(T_i) - 5\prod_{i=1}^{k} t(T_i, v_i) \\ &= 8s(T_j)\prod_{\substack{i=1\\ i\neq j}}^{k} s(T_i) - 5(s(T_j) - x(T_j, v_j))\prod_{\substack{i=1\\ i\neq j}}^{k} t(T_i, v_i) \\ &= \left(8\prod_{\substack{i=1\\ i\neq j}}^{k} s(T_i) - 5\prod_{\substack{i=1\\ i\neq j}}^{k} t(T_i, v_i)\right) s(T_j) + 5\prod_{\substack{i=1\\ i\neq j}}^{k} t(T_i, v_i) \cdot x(T_j, v_j) \end{aligned}$$

for all j, and that $8\prod_{\substack{i=1\\ i\neq j}}^{k} s(T_i) - 5\prod_{\substack{i=1\\ i\neq j}}^{k} t(T_i, v_i) > 0$. Hence, by the induction hypothesis, every branch of a tree for which $s(T)$ is minimal has to be a star, rooted at its center (or a single vertex). The same way of reasoning works for the tree for which $t(T, v)$ is minimal. Finally, the branches of a tree for which $x(T, v)$ is minimal have to be stars (or possibly single vertices), each rooted at one of its leaves.

For $s(T)$, the argument is easy now: without loss of generality, we may assume that v is a leaf, and the claim readily follows.

For $t(T, v)$, we have to minimize the expression

$$8\prod_{i=1}^{k}(8 \cdot 3^{n_i-1} - 5 \cdot 2^{n_i-1}) - 6\prod_{i=1}^{k}(8 \cdot 3^{n_i-1} - 6 \cdot 2^{n_i-1})$$

subject to the condition $\sum_{i=1}^{k} n_i = n - 1$ (here, n_i is the number of vertices in the i-th branch). First, suppose that $k \geq 2$, consider two branches l, m, and set $n_l + n_m = p$. We write A and B for

$$\prod_{\substack{i=1\\ i\neq l,m}}^{k}(8 \cdot 3^{n_i-1} - 5 \cdot 2^{n_i-1}) \quad \text{and} \quad \prod_{\substack{i=1\\ i\neq l,m}}^{k}(8 \cdot 3^{n_i-1} - 6 \cdot 2^{n_i-1})$$

respectively and note that $A \geq B$. Now, we have

$$\begin{aligned} &8\prod_{i=1}^{k}(8 \cdot 3^{n_i-1} - 5 \cdot 2^{n_i-1}) - 6\prod_{i=1}^{k}(8 \cdot 3^{n_i-1} - 6 \cdot 2^{n_i-1}) \\ &\quad = 8A(8 \cdot 3^{n_l-1} - 5 \cdot 2^{n_l-1})(8 \cdot 3^{n_m-1} - 5 \cdot 2^{n_m-1}) \\ &\qquad - 6B(8 \cdot 3^{n_l-1} - 6 \cdot 2^{n_l-1})(8 \cdot 3^{n_m-1} - 6 \cdot 2^{n_m-1}) \\ &\quad = (25A - 27B) \cdot 2^{p+1} + 128(4A - 3B) \cdot 3^{p-2} \\ &\qquad - \frac{16(10A - 9B)}{3}\left(3^p\left(\frac{2}{3}\right)^{n_l} + 2^p\left(\frac{3}{2}\right)^{n_l}\right), \end{aligned}$$

and this expression is a concave function in n_l. Hence, the minimum is attained at the borders, namely if $n_l = 1$ or $n_l = p - 1$. But this means that the overall minimum can only be attained if all but one n_l equals 1. Thus let

$n_1 = n_2 = \cdots = n_{k-1} = 1$ and $n_k = n - k$. Then we have to minimize

$$8 \cdot 3^{k-1}(8 \cdot 3^{n-k-1} - 5 \cdot 2^{n-k-1}) - 6 \cdot 2^{k-1}(8 \cdot 3^{n-k-1} - 6 \cdot 2^{n-k-1})$$
$$= 64 \cdot 3^{n-2} + 9 \cdot 2^n - 8 \cdot 3^n \left(\frac{2}{3}\right)^k - \frac{20}{3} \cdot 2^n \left(\frac{3}{2}\right)^k,$$

which is again a concave function (in k). Comparing the values at the borders, we see that the minimum is attained for $k = 1$, which corresponds to a star, v being one of its leaves.

Finally, in order to minimize $x(T, v)$, we have to minimize

$$\prod_{i=1}^{k} f(n_i),$$

where $\sum_{i=1}^{k} n_i = n-1$ and $f(x) = 16\cdot 3^{x-2} - 2^x$ for $x > 1$ and $f(1) = 2$. However, the simple inequality $f(x) \geq 2^x$ holds, with equality if and only if $x = 1$. Therefore, the minimum is attained if and only if $n_1 = n_2 = \cdots = 1$, and the minimal value is 2^{n-1}. This finishes the induction and hence the whole argument. $\square$

To conclude this section, we remark that the recursion for $s(T)$ and $t(T, v)$ can easily be translated to the world of generating functions (compare [21]). For instance, if one wants to determine the average solvability number of a rooted ordered tree, the following functional equations for the generating functions $S(x) = \sum_T s(T)x^{|T|}$ and $T(x) = \sum_T t(T, v)x^{|T|}$ hold:

$$S(x) = \frac{8x}{1 - S(x)} - \frac{5x}{1 - T(x)},$$
$$T(x) = \frac{8x}{1 - S(x)} - \frac{6x}{1 - T(x)}.$$

Making use of Gröbner bases and the power of a computer algebra system, one can reduce this system to a single equation for the generating function S:

$$6S(x)^4 - 17S(x)^3 + (31x + 16)S(x)^2 - (46x + 5)S(x) + 64x^2 + 15x = 0.$$

It is well known (see [5,7]) that the smallest singularity of a function that is given by such a polynomial equation of the form $F(S, x) = 0$ can be found by computing the common zeros of $F(S, x) = 0$ and $F_S(S, x) = 0$ (the only other potential singularities are zeros of the coefficient of the highest power of S in F, but there are no such zeros in our case). This leads to a single polynomial equation for x:

$$x^3(1 + 3x)(1 - 293x + 4232x^2) = 0.$$

Moreover, we can easily give a crude estimate for the radius of convergence of S: since there are $\frac{1}{n}\binom{2n-2}{n-1}$ rooted ordered trees with n vertices, we have

$$3^n \cdot \frac{1}{n}\binom{2n-2}{n-1} \leq s_n \leq 4^n \cdot \frac{1}{n}\binom{2n-2}{n-1},$$

where s_n is the coefficient of x^n in S. Therefore, the radius of convergence lies between $\frac{1}{16}$ and $\frac{1}{12}$, and so the dominant singularity has to be $\rho = \frac{293+41\sqrt{41}}{8464}$. The expansion of S around this singularity is

$$S(x) = \frac{125 - 7\sqrt{41}}{184} - \sqrt{\frac{3567 + 523\sqrt{41}}{30176}}(1 - x/\rho)^{1/2} + \cdots.$$

By a standard singularity analysis (see [11,12]), we find that the average solvability number of a rooted ordered tree on n vertices is asymptotically

$$\sqrt{\frac{3567 + 523\sqrt{41}}{7544}} \cdot \left(\frac{293 - 41\sqrt{41}}{8}\right)^n.$$

4. Random graphs

In this section, we will show that the expected value of the solvability number $s(G)$ is of order 4^n for a random graph in $\mathcal{G}(n, \frac{1}{2})$ (i.e., each edge is inserted with probability $\frac{1}{2}$), so that the "typical" value of $s(G)$ is pretty close to its maximum. In particular, the following theorem holds:

Theorem 15. *Let $G \in \mathcal{G}(n, \frac{1}{2})$ be a random graph with vertex set $[n] = \{1, 2, \ldots, n\}$. Then the inequality*

$$\mathbb{E}(s(G)) > \frac{1}{2} \cdot 4^n$$

holds.

Proof. The proof of this theorem essentially follows the approach of Amin, Clark and Slater [1]. First of all, fix a vector $a \in \{0, 1\}^n$, and let A denote the (random) adjacency matrix of G. Note that the number of solutions of the matrix equation $(A + \mathrm{diag}(a))x = 0$ over $\mathbb{F}_2$ is exactly $X = 2^{n-\mathrm{rk}(A+\mathrm{diag}(a))}$. We will calculate the expected value of this random variable rather than that of $2^n X^{-1}$ which we are actually interested in. To this end, we determine the probability that a set S is a solution for the instance $(a, 0)$. We write p_S for this probability and obtain

$$\mathbb{E}(X) = \sum_{S \subseteq [n]} p_S.$$

Let s_1, s_2 be the number of vertices $v \in S$ such that $a_v = 1$ and $a_v = 0$ respectively, and let S_1, S_2 be the corresponding sets. Furthermore, set $s = |S|$. S can only be a solution if

- every vertex in S_1 has an odd number of neighbors in S,
- every vertex in S_2 has an even number of neighbors in S,
- every vertex in $[n] \setminus S$ has an even number of neighbors in S.

If G_S is the restriction of G on S, the first two statements are equivalent to the property that S_1 is the set of vertices of odd degree in G_S. By the following lemma, the probability for this is $2^{\binom{s-1}{2}}/2^{\binom{s}{2}} = 2^{1-s}$ if $S \neq \emptyset$ and if s_1 is even:

Lemma 16 (*Read and Robinson [18]*). *Let $U \subseteq V$ with $|V| = n$. If $|U|$ is even, the number of simple graphs on V where U is the set of vertices having odd degree is $2^{\binom{n-1}{2}}$.*

Furthermore, the probability that the number of neighbors in S is even equals $\frac{1}{2}$ for every vertex in $[n] \setminus S$, as long as $S \neq \emptyset$. By independence, we thus have $p_S = 2^{1-s} 2^{-(n-s)} = 2^{1-n}$.

$S = \emptyset$ is always a solution, so $p_S = 1$ in this case. So if k is the number of 1's in a, we obtain

$$\mathbb{E}(X) = \sum_{\substack{s_1=0 \\ s_1 \text{ even}}}^{k} \binom{k}{s_1} \sum_{s_2=0}^{n-k} \binom{n-k}{s_2} 2^{1-n} + (1 - 2^{1-n}) = \begin{cases} 2 - 2^{1-n} & k \neq 0, \\ 3 - 2^{1-n} & k = 0. \end{cases}$$

By Jensen's inequality,

$$2^n \mathbb{E}(X^{-1}) \geq \begin{cases} \dfrac{2^n}{2 - 2^{1-n}} & k \neq 0, \\ \dfrac{2^n}{3 - 2^{1-n}} & k = 0. \end{cases}$$

Summing over all vectors a finally shows that

$$\mathbb{E}(s(G)) \geq \frac{2^n(2^n - 1)}{2 - 2^{1-n}} + \frac{2^n}{3 - 2^{1-n}} > \frac{1}{2} \cdot 4^n. \quad \square$$

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